APPLIED MATHEMATICS FOR ENGINEERS
MIDTERM 2

Code : *MAT 210* Acad.Year : *2018-19* Semester : *Spring* Date : *28.04.2019* Time : *9:40* Duration : *110 min*	Last Name : Name : Student ID : Signature :	# :
	9 QUESTIONS ON 5 PAGES TOTAL 100 POINTS	

P1. (20)	P2. (20)	P3. (20)	P4. (20)	P5. (20)		Total. (100)

1. *(2× 10=20pts)* Indicate whether a given statement is **TRUE** or **FALSE** by circling your answer. No explanations are required.

TRUE / FALSE In a truss system, if Bw = f then f is a mechanism.

$B\underline{w} = f$ for f balanced $\qquad B^T f = 0$ for f a mechanism

TRUE / FALSE In a truss system, if Af = 0 then f is a balanced force.

See above, recall $A = B^T$

TRUE / FALSE A truss system is stable if the force balance matrix B has no nullspace.

Stable : no mechanisms $\iff$ B^T has no nullspace

TRUE / FALSE A truss system has no extra bars if the LU decomposition of the force balance matrix B has a pivot in each column.

columns without pivot $\iff$ unnecessary bars

TRUE / FALSE If the LU decomposition of the force balance matrix B of a truss system has #(pivots) < 2· #(nodes) then the system must be unstable.

#pivots < 2 #(nodes) = #(columns in B^T) $\Rightarrow$ nullspace of B^T is non-zero $\Rightarrow$ mechanisms, so unstable

TRUE / FALSE If a spring system has no outside connection (no ceiling, no floor) then a vector f of forces on its nodes is balanced if all forces are in the same direction.

Balanced force vector must sum to 0 : impossible if forces are in same direction!

TRUE / FALSE The stiffness matrix is symmetric – i.e. $K = K^T$.

TRUE / FALSE $A^T w = f$ where A is the elongation matrix, w is internal force in bars and f is resulting force on nodes.

$A^T = B$ so this is the force-balance equation

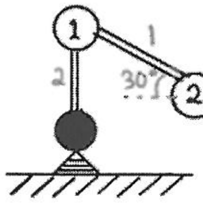
TRUE / FALSE The sum of all of the numbers in a Markov matrix is 1.

The sum of each column is 1

TRUE / FALSE A Markov matrix can have negative eigenvalues.

eigenvalues are $-1 \le \lambda \le 1$

2. *(2+4+4=10pts)* Consider the truss system to the right.



(A) Write the force balance matrix.

$$B = \begin{array}{c} \text{bar 1} \quad \text{bar 2} \\ \left[\begin{array}{c|c} -\cos 30^\circ & 0 \\ \sin 30^\circ & 1 \\ \hline \cos 30^\circ & 0 \\ -\sin 30^\circ & 0 \end{array}\right] \end{array} \begin{array}{l} x \\ y \\ x \\ y \end{array} \begin{array}{l} \text{node } 1 \\ \\ \text{node } 2 \\ \\ \end{array} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 1 \\ \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

(B) Show that $\mathbf{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1/\sqrt{3} \end{bmatrix}$ is a balanced force.

$\underline{f}$ balanced $\Leftrightarrow$ $\underline{f} = B\underline{w}$ has solution $\underline{w}$:

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & 0 \\ 1/2 & 1 \\ \sqrt{3}/2 & 0 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Leftrightarrow \begin{cases} 1: 1 = -\frac{\sqrt{3}}{2} w_1 \\ \quad \Rightarrow w_1 = -\frac{2}{\sqrt{3}} \\ 2: \frac{1}{2} \cdot -\frac{2}{\sqrt{3}} + w_2 = 0 \\ \quad \Rightarrow w_2 = 1/\sqrt{3} \\ 3: -1 = \frac{\sqrt{3}}{2} \cdot -\frac{2}{\sqrt{3}} \checkmark \\ 4: 1/\sqrt{3} = -\frac{1}{2} \cdot -\frac{2}{\sqrt{3}} \checkmark \end{cases}$$

So: $\underline{f} = B\underline{w}$ has solution $\underline{w} = \begin{bmatrix} -2/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

(C) Show that $\mathbf{f} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ \sqrt{3} \end{bmatrix}$ is a mechanism.

$\underline{f}$ is mechanism $\Leftrightarrow$ $\underline{f} \in$ nullsp(B^T) $\Leftrightarrow$ $B^T \underline{f} = \underline{0}$

$$B^T \underline{f} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} + 0 + \sqrt{3} - \frac{\sqrt{3}}{2} \\ 0 + 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

3. *(3+3+4=10pts)* Reducing a matrix (with row swaps and scaling) gives

$$A = \begin{bmatrix} 4 & -8 & 4 & 0 & 0 \\ 2 & -1 & 2 & -6 & 3 \\ 4 & -8 & 4 & 0 & 0 \\ -1 & 0 & -1 & 4 & -3 \end{bmatrix} \longrightarrow \begin{array}{c} 1 \\ 2 \\ 3 \\ \\ \end{array} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

pivots

Give bases for the following.

(A) Row space of A.

rowsp(A) = span (rows with pivots in U)

= span([1 -2 1 0 0], [0 1 0 -2 1], [0 0 0 0 1])

(B) Column space of A.

colsp(A) = span (columns in A corresponding to pivot columns in U)

$$= \text{span}\left(\begin{bmatrix} 4 \\ 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -8 \\ -1 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ -3 \end{bmatrix} \right)$$

(C) Null space of A.

$$\text{nullsp}(A) = \text{nullsp}(U) = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) \begin{array}{l} v \\ w \\ x \\ y \\ z \end{array}$$

$v \leftarrow$ row 1: $v = 2w - x$

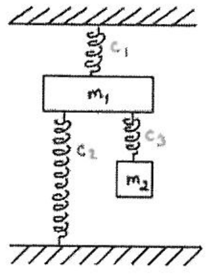
$w \leftarrow$ row 2: $w = 2y - z$

$x \leftarrow$ 3rd col. pivotless in U } fill in first

$y \leftarrow$ 4th col. pivotless in U } fill in first

$z \leftarrow$ row 3: $z = 0$

4. *(8pts)* Consider the spring system pictured to the right with spring constants $c_1 = 2$, $c_2 = 1$, $c_3 = 3$.
(Let down be the positive direction.)



Write the stiffness matrix and compute the equilibrium displacement of the masses if they are acted on by external forces $f_1 = 9$ and $f_2 = -6$.

⊛ Stiffness matrix

$$K = \begin{array}{cc} m_1 & m_2 \\ \left[\begin{array}{c|c} 2+1+3 & -3 \\ \hline -3 & 3 \end{array}\right] & \begin{array}{c} m_1 \\ m_2 \end{array} \end{array} = \begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix}$$

⊛ $$\begin{bmatrix} 9 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \iff \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{6\cdot 3 - -3 \cdot -3} \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 9 \\ -6 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 27-18 \\ 27-36 \end{bmatrix} = \begin{bmatrix} 9/9 \\ -9/9 \end{bmatrix}$$

displacements of masses

So: $\begin{cases} m_1 \text{ moves down } 1 \text{ unit} \\ m_2 \text{ moves up } -1 \text{ unit.} \end{cases}$

5. *(6+6=12pts)* Suppose an oscillating spring system with three masses and multiple springs has the following eigenvalues and eigenvectors of $M^{-1}K$:

$$\lambda_1 = 0, \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \qquad \lambda_2 = 2, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \qquad \lambda_3 = 3, \quad \mathbf{v_3} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(A) Give the general solution for the mass displacement function $\mathbf{u}(t)$.

this is an unbalanced system (no outside connection)!

$$\underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = p_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + q_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$$
$$+ p_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cos(\sqrt{2}t) + q_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \sin(\sqrt{2}t)$$
$$+ p_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cos(\sqrt{3}t) + q_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \sin(\sqrt{3}t)$$

here $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{R}$ are constants.

(B) Give the solution if the initial displacement and velocity are $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{u}'(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \underline{u}(0) = p_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + p_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + p_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Longrightarrow \begin{cases} p_1 = 0 \\ p_2 = 1 \\ p_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underline{u}'(0) = q_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + q_2 \sqrt{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + q_3 \sqrt{3} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Longrightarrow \begin{cases} q_1 = 1 \\ q_2 = 0 \\ q_3 = 0 \end{cases}$$

So: $$\underline{u}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cos(\sqrt{2}t).$$

6. *(3+3=6pts)* Suppose a spring system with three masses oscillates at a fundamental mode corresponding to the following eigenvalue and eigenvector of $M^{-1}K$:

$$\text{eigenvalue } \lambda = 4 \text{ and eigenvector } \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

(A) If mass 2 oscillates with amplitude 6, then what is the amplitude of oscillation of mass 3?

ratios between amplitudes:

$[a_1 : a_2 : a_3] = [1 : 2 : 4]$

So if $a_2 = 6$ then $a_3 = 12$:

mass 3 oscillates with amplitude 12.

(B) If mass 3 is at maximum height at time $t = 4$ then when will it next be at maximum height?

next max @ 4 + (period of oscillation)

$$= 4 + \underbrace{\frac{2\pi}{\omega}}_{\text{frequency}} = 4 + \frac{2\pi}{\sqrt{\lambda}} = 4 + \pi$$

7. *(2+2+4+6=14pts)* The transition matrix K for a Markov system has the following eigenvalues and eigenvectors.

$$\lambda_1 = 1/2, \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \qquad \lambda_2 = -1/3, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \qquad \lambda_3 = 1, \quad \mathbf{v_3} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

If the system begins with 40 people all in state 3 then find the following.

(A) The long-term **probability** vector.

1-eigenvector

long term prob. = stable vector whose coordinates add up to 1:

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{2+1+1} = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/4 \end{bmatrix}$$

(B) The **initial** state vector.

$$\begin{bmatrix} 0 \\ 0 \\ 40 \end{bmatrix}$$

(C) The **stable** state vector.

40 people in this system so:

stable vector whose coordinates add up to 40:

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{40}{2+1+1} = \begin{bmatrix} 20 \\ 10 \\ 10 \end{bmatrix}$$

(D) The nth state vector.

⊛ Split $\begin{bmatrix} 0 \\ 0 \\ 40 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

(SHORTCUT: $x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $y \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ add up to zero. So to keep 40 people, z has to be 10. $\Rightarrow$ $x = -30$ and $y = 10$ then easily follow)

$$\Leftrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 40 \end{bmatrix} \overset{LU}{\Leftrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}^{L} \overbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}^{U} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 40 \end{bmatrix}$$

L^{-1}

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -30 \\ 10 \\ 10 \end{bmatrix} \xleftarrow{U^{-1}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 40 \end{bmatrix}$$

⊛ Apply K^n to the split, remember $K^n v_i = \lambda_i^n v_i$:

$$\text{n-th state} = K^n \begin{bmatrix} 0 \\ 0 \\ 40 \end{bmatrix} = \underbrace{-30}_{x} (1/2)^n \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \underbrace{10}_{y} (-1/3)^n \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \underbrace{10}_{z} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

8. *(12pts)* A Markov system has the following transition matrix. Find the eigenvalues. **(Do not find the eigenvectors.)**

$$K = \begin{bmatrix} 3/4 & 1/2 & 0 \\ 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 3/4 \end{bmatrix}$$

⊛ Working with 4K is easier, and $\underbrace{\phi_i}_{\text{eigenvalue of } 4K} = 4 \underbrace{\lambda_i}_{\text{eigenvalue of } K.}$

⊛ Char. polynomial of 4K:

$$|4K - \phi I| = \begin{vmatrix} 3-\phi & 2 & 0 \\ 0 & 1-\phi & 1 \\ 1 & 1 & 3-\phi \end{vmatrix} = (3-\phi)\begin{vmatrix} 1-\phi & 1 \\ 1 & 3-\phi \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 1 & 3-\phi \end{vmatrix} + 0\begin{vmatrix} 0 & 1-\phi \\ 1 & 1 \end{vmatrix}$$

$$= (3-\phi)\left[(1-\phi)(3-\phi)-1\right] - 2(0-1) = (3-\phi)(\phi^2-4\phi+2)+2 = -\phi^3+7\phi^2-14\phi+8$$

⊛ K is transition matrix, $\lambda = 1$ highly likely → does 4K have $\phi = 4$ as eigenvalue?

$$|4K-\phi I|\Big|_{\phi=4} = -4^3 + 7\cdot 4^2 - 14\cdot 4 + 8 = \cancel{-4^3} + \cancel{4\cdot 4^2} + 3\cdot 4^2 - 56 + 8 = 48 - 48 = 0 \checkmark$$

⊛ $\phi = 4$ is a root of $|4K - \phi I|$, factorise to find others:

$$\begin{array}{r} -\phi^2+3\phi-2 \\ \phi-4 \,\overline{\big)\, -\phi^3+7\phi^2-14\phi+8} \\ \underline{-\phi^3+4\phi^2} \quad\quad\quad \\ 3\phi^2-14\phi \quad \\ \underline{3\phi^2-12\phi} \quad \\ -2\phi+8 \\ \underline{-2\phi+8} \\ 0 \end{array}$$

So: $|4K-\phi I| = (\phi-4)(-\phi^2+3\phi-2) = -(\phi-4)(\phi^2-3\phi+2)$
$= -(\phi-4)(\phi-2)(\phi-1)$

→ eigenvalues of 4K are $\phi = 4, 2, 1$ $\div 4$

→ eigenvalues of K are $\boxed{\lambda = 1, 1/2, 1/4}$

9. *(4+4=8pts)* The following parts involve the matrix. $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

(A) Show that $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is **not** an eigenvector.

There should be <u>no</u> λ s.t. $A\underline{v} = \lambda \underline{v}$.

$$A\underline{v} = \begin{bmatrix} 0+1-1 \\ 2-1+1 \\ 0+2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \stackrel{?}{=} \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{cases} \lambda = 0 \\ \lambda = 2 \\ \lambda = 3 \end{cases}$$

Impossible!

(B) Show that $\lambda = 1$ is **not** an eigenvalue.

$A\underline{v} = 1\underline{v}$ should have no nonzero solutions

⇕

$$(A-I)\underline{v} = 0 \Leftrightarrow \begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -2 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \left\{\begin{array}{l} ③:\ 2y = 0 \Rightarrow y = 0 \\ ①:\ -x + \underbrace{0}_{y} - z = 0 \Rightarrow x = -z \\ ②:\ 2\cdot\underbrace{-z}_{x} + \underbrace{0}_{y} + z = 0 \Rightarrow z = 0 \end{array}\right\} \Rightarrow x = 0$$

Only sol. to $A\underline{v} = \underline{v}$ is $\underline{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

→ 1 is not an eigenvalue!